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Exact multiplicity of solutions and S-shaped bifurcation curves for the p-Laplacian perturbed Gelfand problem in one space variable ☆

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Abstract

We study exact multiplicity of positive solutions and the bifurcation curve of the p-Laplacian perturbed Gelfand problem from combustion theory

$$\begin{cases} (\varphi_p(u'(x)))' + \lambda \exp\left(\dfrac{au}{a+u}\right) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases}$$

where $p > 1$, $\varphi_p(y) = |y|^{p-2}y$, $(\varphi_p(u'))'$ is the one-dimensional p-Laplacian, $\lambda > 0$ is the Frank–Kamenetskii parameter, $u(x)$ is the dimensionless temperature, and the reaction term $f(u) = \exp(\frac{au}{a+u})$ is the temperature dependence obeying the Arrhenius reaction-rate law. We find explicitly $\tilde{a} = \tilde{a}(p) > 0$ such that, if the activation energy $a \geqslant \tilde{a}$, then the bifurcation curve is S-shaped in the $(\lambda, \|u\|_\infty)$-plane. More precisely, there exist $0 < \lambda_* < \lambda^* < \infty$ such that the problem has exactly three positive solutions for $\lambda_* < \lambda < \lambda^*$, exactly two positive solutions for $\lambda = \lambda_*$ and $\lambda = \lambda^*$, and a unique positive solution for $0 < \lambda < \lambda_*$ and $\lambda^* < \lambda < \infty$.

Keywords: Exact multiplicity; Positive solution; S-shaped bifurcation curve; p-Laplacian; Perturbed Gelfand problem; Time map

1. Introduction

We study exact multiplicity of positive solutions and the bifurcation curve of the p-Laplacian perturbed Gelfand problem

$$\begin{cases} (\varphi_p(u'(x)))' + \lambda \exp\left(\dfrac{au}{a+u}\right) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases} \tag{1.1}$$

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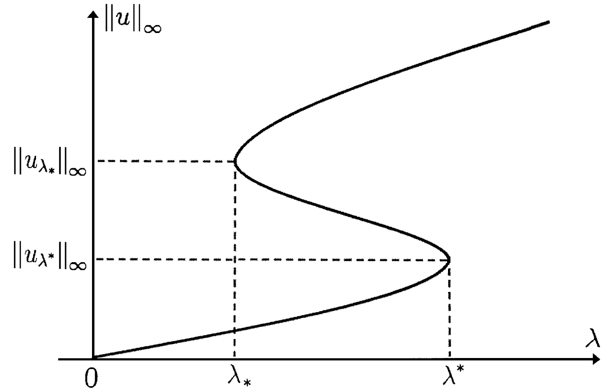

Fig. 1. S-shaped bifurcation curve $\bar{S}$ of (1.1) for $p > 1$ and $a \geqslant \tilde{a}$.

where $p > 1$, $\varphi_p(y) = |y|^{p-2}y$, $(\varphi_p(u'))'$ is the one-dimensional p-Laplacian. This is the one-dimensional case of a problem arising in the study of (steady state) solid fuel ignition models in thermal combustion theory, and it was discussed in [1] and many references cited within for the case when $p = 2$ (Laplacian case). See also [7, Section 4]. In this context, the quantity p is a characteristic of the medium, $\lambda > 0$ is the Frank–Kamenetskii parameter, $u(x)$ is the dimensionless temperature, the reaction term $f(u) = \exp(\frac{au}{a+u})$ is the temperature dependence obeying the Arrhenius reaction-rate law, and $a > 0$ is the activation energy.

It is known that, for any given $\alpha > 0$, there exists a unique $\lambda = \lambda(\alpha) > 0$ such that (1.1) admits a unique positive solution u with $\|u\|_\infty = \alpha$. We define the bifurcation curve of (1.1)

$$\bar{S} = \big\{\big(\lambda, \|u\|_\infty\big):\ \lambda > 0 \text{ and } u \text{ is a positive solution of (1.1)}\big\}.$$

In Theorem 2.1 stated below, which is the main result in this paper, we find explicitly $\tilde{a} = \tilde{a}(p) > 0$ such that, if the activation energy $a \geqslant \tilde{a}$, then the bifurcation curve $\bar{S}$ is S-shaped in the $(\lambda, \|u\|_\infty)$-plane; that is, $\bar{S}$ has *exactly two* turning points at some points $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ and $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ such that

(i) $\lambda_* < \lambda^*$ and $\|u_{\lambda^*}\|_\infty < \|u_{\lambda_*}\|_\infty$,
(ii) at $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ the bifurcation curve $\bar{S}$ turns to the left,
(iii) at $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ the bifurcation curve $\bar{S}$ turns to the right.

More precisely, problem (1.1) has exactly three positive solutions for $\lambda_* < \lambda < \lambda^*$, exactly two positive solutions for $\lambda = \lambda_*$ and $\lambda = \lambda^*$, and a unique positive solution for $0 < \lambda < \lambda_*$ and $\lambda^* < \lambda < \infty$. See Fig. 1.

We recall some results on S-shaped bifurcation curves for bifurcation problems related to (1.1). When $p = 2$, problem (1.1) reduces to

$$\begin{cases} u''(x) + \lambda \exp\left(\dfrac{au}{a+u}\right) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0. \end{cases} \tag{1.2}$$

It is well known that, if $0 < a \leqslant 4$, the bifurcation curve $\bar{S}$ for (1.2) is a monotone curve in the $(\lambda, \|u\|_\infty)$-plane, see, e.g., [2, p. 482]. That is, $\bar{S}$ has no turning point. In [12], using the quadrature method, Wang proved the next S-shaped bifurcation curve theorem for (1.2) for $a > a^* \approx 4.4967$ with some constant a^* defined in [12, Eq. (2.21)].

Theorem 1.1. *(See Wang [12], Fig. 1.) Let* $p = 2$. *Consider* (1.2). *There exists* $a^* \approx 4.4967$ *such that if* $a > a^*$, *then*

$$\lim_{\alpha \to 0^+} \lambda(\alpha) = 0, \qquad \lim_{\alpha \to \infty} \lambda(\alpha) = \infty,$$

and the bifurcation curve $\bar{S}$ *is S-shaped in the* $(\lambda, \|u\|_\infty)$*-plane. More precisely, there exist* $0 < \lambda_* < \lambda^* < \infty$ *such that* (1.2) *has exactly three positive solutions for* $\lambda_* < \lambda < \lambda^*$, *exactly two positive solutions for* $\lambda = \lambda_*$ *and* $\lambda = \lambda^*$, *and a unique positive solution for* $0 < \lambda < \lambda_*$ *and* $\lambda^* < \lambda < \infty$.

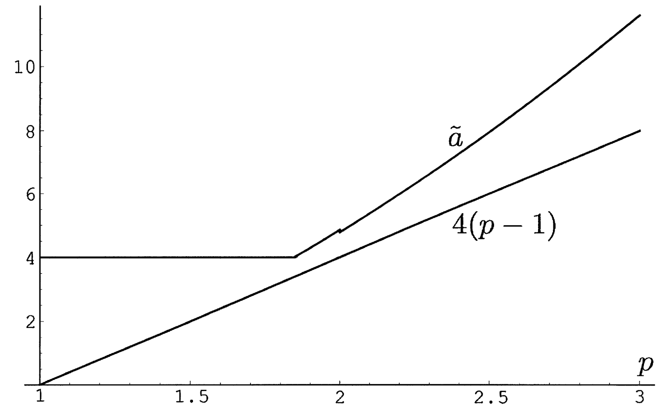

Fig. 2. Graphs of $\tilde{a} > 4(p-1)$ for $1 < p \leqslant 3$.

Note that the result in Theorem 1.1 was improved by Korman and Li [8, Theorem 3.1] for $a > a^{**} \approx 4.35$ by applying a bifurcation theorem of Crandall and Rabinowitz [4]. When $\Omega = B_1$ is the unit ball in $\mathbb{R}^2$, Du and Lou [6] proved that the bifurcation curve $\{(\lambda, \|u\|_\infty)\}$ of the perturbed Gelfand problem

$$\begin{cases} \Delta u + \lambda \exp\left(\dfrac{au}{a+u}\right) = 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1 \end{cases}$$

is S-shaped if a is large enough.

In [13], Wang studied problem (1.2) with nonlinearity $f(u) = \exp(\frac{au}{a+u})$ replaced by $\tilde{f}(u) = (1 + u/a)^m \exp(\frac{au}{a+u})$, in which $\tilde{f}$ is the temperature dependence of mth ($m < 1$) order reaction rate obeying the *general* Arrhenius reaction-rate law. Du [5] extended the results of [13] for the one-dimensional case to cover both dimensions one and two, and extended the results of [6] for the special case $m = 0$ to the general case $0 \leqslant m < 1$.

Recently, Ramaswamy and Shivaji [11, Section 4] studied existence of multiple positive solutions of the p-Laplacian perturbed Gelfand problem

$$\begin{cases} \Delta_p u + \lambda \exp\left(\dfrac{au}{a+u}\right) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $\lambda > 0$, and Ω is a general bounded domain in $\mathbb{R}^N$, $N \geqslant 2$ with $\partial\Omega$ of class C^2 and connected. (If $N = 1$, they assumed that Ω is a bounded open interval.) They showed that problem (1.3) has *at least three* positive solutions for a certain range of λ if a is large enough by applying the sub–super solution techniques. Note that it is well known that a necessary condition for multiplicity for (1.3) and also for (1.1) is $a > 4(p-1)$, see, e.g., [11, Section 4]. See also Jacobsen and Schmitt [7, Section 4] for precise existence and multiplicity results for radial solutions of the p-Laplacian Gelfand problem

$$\begin{cases} \Delta_p u + \lambda \exp(u) = 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases}$$

where B_1 is the unit ball in $\mathbb{R}^N$ ($N \geqslant 1$).

The paper is organized as follows. Section 2 contains statements of Theorem 2.1, Lemmas 2.2–2.4 used to prove Theorem 2.1, and the proof of Theorem 2.1. Section 3 contains proofs of Lemmas 2.2–2.4.

2. Main result

Theorem 2.1. *(See Figs. 1 and 2.) Let* $p > 1$. *Consider* (1.1). *If*

$$a \geqslant \tilde{a} \equiv \begin{cases} \max\{4, (4p-2)(\ln p) + 0.7\} & \text{if } 1 < p < 2, \\ (4p-2)(\ln p) + \eta & \text{if } p \geqslant 2, \end{cases} \tag{2.1}$$

where

$$\eta \approx 0.622$$

is the positive zero of

$$M(u) \equiv 4u \exp(u/2) - u - 4\ln 2$$

satisfying $M(u) < 0$ for $0 < u < \eta$, then the bifurcation curve $\bar{S}$ is S-shaped in the $(\lambda, \|u\|_\infty)$-plane. More precisely, there exist $0 < \lambda_* < \lambda^* < \infty$ such that (1.1) *has exactly three positive solutions for $\lambda_* < \lambda < \lambda^*$, exactly two positive solutions for $\lambda = \lambda_*$ and $\lambda = \lambda^*$, and a unique positive solution for $0 < \lambda < \lambda_*$ and $\lambda^* < \lambda < \infty$.*

Remark 1. In (2.1),

(i) For $1 < p < 2$,

$$\tilde{a} \equiv \max\{4, (4p-2)(\ln p) + 0.7\} = \begin{cases} 4 & \text{for } 1 < p \leqslant p^*, \\ (4p-2)(\ln p) + 0.7 & \text{for } p^* < p < 2, \end{cases}$$

where we define $p^* \approx 1.846$ is the unique root of the equation $(4p-2)(\ln p) + 0.7 = 4$ in the interval $(1, 2)$.

(ii) For $p = 2$, $\tilde{a} = 6(\ln 2) + \eta \approx 4.781 > 4.4967 \approx a^*$, in which a^* stated in Theorem 1.1, is a lower bound of activation energy a such that the bifurcation curve $\bar{S}$ is S-shaped in the $(\lambda, \|u\|_\infty)$-plane. Hence Theorem 2.1 contains no new result for the case $p = 2$.

(iii) For $p > 1$,

$$\tilde{a} > 4(p-1) \tag{2.2}$$

by a numerical simulation given in Fig. 2.

We first consider the bifurcation curve of positive solutions of the p-Laplacian Dirichlet problem

$$\begin{cases} \left(\varphi_p\left(u'(x)\right)\right)' + \lambda f\left(u(x)\right) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases} \tag{2.3}$$

where $f \in C[0,\infty) \cap C^2(0,\infty)$, and $\lambda > 0$ is a bifurcation parameter. We define

$$F(u) = \int_0^u f(t)\,dt \quad \text{and} \quad \theta(u) = pF(u) - uf(u), \tag{2.4}$$

and assume that function f satisfies the following hypotheses (H1)–(H3):

(H1) $f(u) > 0$ for $u \in [0,\infty)$, and $m_\infty \equiv \lim_{u\to\infty} f(u)/u^{p-1} = 0$.

(H2) There exist numbers $0 < C_1 < D_1 < \gamma < C_2 < D_2 < \infty$ such that

$$\theta(D_1) = \theta(D_2) = \theta'(C_1) = \theta'(C_2) = 0, \tag{2.5}$$

and either

$$\theta''(u) = (p-2)f'(u) - uf''(u) \begin{cases} < 0 & \text{on } (0,\gamma), \\ = 0 & \text{for } u = \gamma, \\ > 0 & \text{on } (\gamma,\infty), \end{cases} \tag{2.6}$$

or there exist a number $\tilde{\gamma} \in (0, C_1)$ such that

$$\theta''(u) = (p-2)f'(u) - uf''(u) \begin{cases} > 0 & \text{on } (0, \tilde{\gamma}), \\ = 0 & \text{for } u = \tilde{\gamma}, \\ < 0 & \text{on } (\tilde{\gamma}, \gamma), \\ = 0 & \text{for } u = \gamma, \\ > 0 & \text{on } (\gamma, \infty), \end{cases} \tag{2.7}$$

and $\theta(\tilde{\gamma}) - \tilde{\gamma}\theta'(\tilde{\gamma}) - \theta(C_2) \geqslant 0$.

(H3) $uf'(u)/f(u) \geqslant -1/(p+1)$ on $(0, C_1)$ and $uf'(u)/f(u)$ is increasing on (C_1, D_1).

The time map formula which we apply to study p-Laplacian problem (2.3) takes the form as follows:

$$T(\alpha) \equiv \left(\frac{p-1}{p}\right)^{1/p} \int_0^\alpha \frac{1}{[F(\alpha) - F(u)]^{1/p}}\, du = \lambda^{1/p} \quad \text{for } 0 < \alpha < \infty, \tag{2.8}$$

see, e.g., [3, Lemmas 2.1 and 2.2] and [10, Lemma 2.4] for the derivation of the time map formula $T(\alpha)$ for problem (2.3). So positive solutions u of (2.3) correspond to $\|u\|_\infty = \alpha$ and $T(\alpha) = \lambda^{1/p}$. Thus to study the number of positive solutions of (2.3) is equivalent to study the shape of the time map $T(\alpha)$ on $(0, \infty)$.

To prove Theorem 2.1, we need the next Lemmas 2.2–2.4 whose proofs are given in the next section.

Lemma 2.2. *Let $p > 1$. Consider* (2.3)*. Suppose $f \in C[0, \infty) \cap C^2(0, \infty)$ satisfies* (H1)–(H3)*, then*

$$\lim_{\alpha \to 0^+} T(\alpha) = 0, \qquad \lim_{\alpha \to \infty} T(\alpha) = \infty, \tag{2.9}$$

and $T(\alpha)$ has exactly two positive critical points, $\alpha^* < \alpha_*$, on $(0, \infty)$, such that $T(\alpha^*)$ is a local maximum on $(0, \infty)$ and $T(\alpha_*)$ is a local minimum on $(0, \infty)$. More precisely, there exist

$$0 < \lambda_* \equiv \big(T(\alpha_*)\big)^p < \lambda^* \equiv \big(T\big(\alpha^*\big)\big)^p < \infty$$

such that (2.3) *has exactly three positive solutions for $\lambda_* < \lambda < \lambda^*$, exactly two positive solutions for $\lambda = \lambda_*$ and $\lambda = \lambda^*$, and a unique positive solution for $0 < \lambda < \lambda_*$ and $\lambda^* < \lambda < \infty$.*

Lemma 2.3. *Let $p > 1$. If*

$$a \geqslant \tilde{a} = \begin{cases} \max\{4, (4p-2)(\ln p) + 0.7\} & \text{if } 1 < p < 2, \\ (4p-2)(\ln p) + \eta & \text{if } p \geqslant 2, \end{cases}$$

then

(i) $\theta(a) < 0$.

(ii) *If $1 < p \leqslant 2$, then there exist numbers $0 < C_1 < D_1 < \gamma < C_2 < D_2 < \infty$ such that* (2.5) *and* (2.6) *hold. Moreover,*

$$C_1 = \frac{a}{2(p-1)}\big[a - 2p + 2 - \sqrt{a^2 + 4a - 4pa}\big], \tag{2.10}$$

$$\gamma = \frac{a}{2p}\big[a - 2p + 2 + \sqrt{a^2 + 4a - 4pa + 4}\big], \tag{2.11}$$

$$C_2 = \frac{a}{2(p-1)}\big[a - 2p + 2 + \sqrt{a^2 + 4a - 4pa}\big]. \tag{2.12}$$

(iii) *If $p \geqslant 2$, then there exist numbers $0 < \tilde{\gamma} < C_1 < D_1 < \gamma < C_2 < D_2 < \infty$ such that* (2.5) *and* (2.7) *hold. In addition,*

$$\theta(\tilde{\gamma}) - \tilde{\gamma}\theta'(\tilde{\gamma}) - \theta(C_2) > 0. \tag{2.13}$$

Moreover, C_1, γ, and C_2 are given in (2.10)–(2.12) *respectively, and*

$$\tilde{\gamma} = \frac{a}{2p}\big[a - 2p + 2 - \sqrt{a^2 + 4a - 4pa + 4}\big]. \tag{2.14}$$

Lemma 2.4. *Let $p \geqslant 2$. If $a \geqslant \tilde{a} = (4p-2)(\ln p) + \eta$, then*

(i) $\tilde{\gamma} < \frac{3}{20}a$,
(ii) $C_1 < \frac{43}{100}a$,
(iii) $\gamma > \frac{139}{100}a$.

We are now in a position to prove Theorem 2.1 by applying Lemma 2.2.

Proof of Theorem 2.1. We show that, for $p > 1$, $f(u) = \exp(\frac{au}{a+u})$ satisfies (H1)–(H3) if $a \geqslant \tilde{a}$. The proof is divided into two cases:

(A) $1 < p < 2$, and
(B) $p \geqslant 2$.

Proof of Case (A) $1 < p < 2$. First, for $1 < p < 2$, for $f(u) = \exp(\frac{au}{a+u})$, it is easy to see that $f \in C[0,\infty) \cap C^2(0,\infty)$ satisfies (H1). We then compute that, for $\theta(u) = pF(u) - uf(u)$ in (2.4),

$$\theta'(u) = (p-1)f(u) - uf'(u) = \left[p - 1 - \frac{a^2 u}{(a+u)^2}\right]\exp\left(\frac{au}{a+u}\right) \tag{2.15}$$

and

$$\theta''(u) = (p-2)f'(u) - uf''(u) = \left[\frac{pu^2 + (2pa - a^2 - 2a)u - a^2(2-p)}{(a+u)^4}\right]a^2 \exp\left(\frac{au}{a+u}\right). \tag{2.16}$$

So, it is easy to see that $\theta(0) = 0$, $\theta'(0) = p - 1 > 0$. Also $\lim_{u\to\infty}\theta(u) = \infty$ since $\lim_{u\to\infty}\theta'(u) = (p-1)\exp(a) > 0$. For

$$1 < p < 2 \quad \text{and} \quad a \geqslant \tilde{a} = \max\{4, (4p-2)(\ln p) + 0.7\},$$

by Lemma 2.3(ii), there exist numbers $0 < C_1 < D_1 < \gamma < C_2 < D_2 < \infty$, where C_1, γ, and C_2 are given in (2.10)–(2.12) respectively, such that (2.5) and (2.6) hold. So f satisfies (H2).

We finally prove that f satisfies (H3). It is clear that $uf'(u) = \frac{a^2u}{(a+u)^2}\exp(\frac{au}{a+u}) > 0$ on $(0,\infty)$. So $uf'(u)/f(u) > 0 > -1/(p+1)$ on $(0, C_1)$. In addition, we compute that

$$\left(\frac{uf'(u)}{f(u)}\right)' = \frac{a^2(a-u)}{(a+u)^3} > 0 \quad \text{on } (0,a) \supset (C_1, D_1)$$

since $(0 < C_1 <)\ D_1 < a$ by applying Lemma 2.3(i) and (ii). Hence f satisfies (H3) if $a \geqslant \tilde{a}$.

We summarize above results and we conclude that, for $1 < p < 2$, $f(u) = \exp(\frac{au}{a+u})$ satisfies (H1)–(H3) if $a \geqslant \tilde{a}$. □

Proof of Case (B) $p \geqslant 2$. Parts of the proof of Case (B) $p \geqslant 2$ are similar to, or the same as, those of Case (A) $1 < p < 2$.

First, for $p \geqslant 2$, for $f(u) = \exp(\frac{au}{a+u})$, it is easy to see that $f \in C[0,\infty) \cap C^2(0,\infty)$ satisfies (H1). We then compute that, for $\theta(u) = pF(u) - uf(u)$ in (2.4), we obtain (2.15) and (2.16) for $\theta'(u)$ and $\theta''(u)$, respectively. So it is easy to see that $\theta(0) = 0$, $\theta'(0) = p - 1 > 0$. Also $\lim_{u\to\infty}\theta(u) = \infty$ since $\lim_{u\to\infty}\theta'(u) = (p-1)\exp(a) > 0$. For

$$p \geqslant 2 \quad \text{and} \quad a \geqslant \tilde{a} = (4p-2)(\ln p) + \eta,$$

by Lemma 2.3(iii), there exist numbers $0 < \tilde{\gamma} < C_1 < D_1 < \gamma < C_2 < D_2 < \infty$, where $\tilde{\gamma}$, C_1, γ, and C_2 are given in (2.14), (2.10)–(2.12) respectively, such that (2.5) and (2.7) hold. In addition, $\theta(\tilde{\gamma}) - \tilde{\gamma}\theta'(\tilde{\gamma}) - \theta(C_2) > 0$. So f satisfies (H2).

Finally, applying Lemma 2.3(i) and (iii), we can prove that f satisfies (H3) for $a \geqslant \tilde{a}$ by using the same argument used in the proof of Case (A) for $1 < p < 2$. We conclude that, for $p \geqslant 2$, $f(u) = \exp(\frac{au}{a+u})$ satisfies (H1)–(H3) if $a \geqslant \tilde{a}$. □

By above, it follows from Lemma 2.2 that the bifurcation curve $\bar{S}$ is S-shaped in the $(\lambda, \|u\|_\infty)$-plane. Moreover, there exist $0 < \lambda_* < \lambda^* < \infty$ such that (1.1) has exactly three positive solutions for $\lambda_* < \lambda < \lambda^*$, exactly two positive solutions for $\lambda = \lambda_*$ and $\lambda = \lambda^*$, and a unique positive solution for $0 < \lambda < \lambda_*$ and $\lambda^* < \lambda < \infty$.

The proof of Theorem 2.1 is complete. □

3. Proofs of Lemmas 2.2–2.4

Proof of Lemma 2.2. Eq. (2.9) follows by a slight generalization of [9, Theorems 2.6 and 2.9]. By (2.8) for $T(\alpha)$, we compute that

$$T'(\alpha) = \left(\frac{p-1}{p^{p+1}}\right)^{1/p} \frac{1}{\alpha} \int_0^\alpha \frac{\Delta\theta}{(\Delta F)^{(p+1)/p}}\, du \tag{3.1}$$

and

$$T''(\alpha) = \left(\frac{p-1}{p^{p+1}}\right)^{1/p} \frac{1}{\alpha^2} \int_0^\alpha \frac{-\frac{p+1}{p}(\Delta\theta)(\Delta\tilde{f}) + \Delta F(\Delta\tilde{\theta}')}{(\Delta F)^{(2p+1)/p}}\, du, \tag{3.2}$$

where $\Delta F = F(\alpha) - F(u)$, $\Delta\theta = \theta(\alpha) - \theta(u)$, $\Delta\tilde{f} = \alpha f(\alpha) - u f(u)$, and $\Delta\tilde{\theta}' = \alpha\theta'(\alpha) - u\theta'(u)$. By (3.1) and (3.2), we obtain that

$$\begin{aligned} T''(\alpha) + \frac{p}{\alpha}T'(\alpha) &= \left(\frac{p-1}{p^{p+1}}\right)^{1/p} \frac{1}{\alpha^2} \int_0^\alpha \frac{(\Delta F)(\Delta\phi) + \frac{p+1}{p}(\Delta\theta)^2}{(\Delta F)^{(2p+1)/p}}\, du \\ &\geqslant \left(\frac{p-1}{p^{p+1}}\right)^{1/p} \frac{1}{\alpha^2} \int_0^\alpha \frac{\Delta\phi}{(\Delta F)^{(p+1)/p}}\, du, \end{aligned} \tag{3.3}$$

where $\phi(u) = u\theta'(u) - \theta(u)$ and $\Delta\phi = \phi(\alpha) - \phi(u)$.

We know that $\theta(0) = 0$, $\theta'(0) = (p-1)f(0) > 0$. By (H1) and (H2), we obtain $\lim_{u\to\infty}\theta(u) = \infty$. In addition, there exists a positive number $D_3 \in (D_2, \infty)$ such that

$$\begin{cases} \theta(u) > 0 \quad \text{on } (0, D_1), \quad \theta(u) < 0 \quad \text{on } (D_1, D_2), \\ 0 < \theta(u) < \theta(C_1) \quad \text{on } (D_2, D_3), \quad \theta(u) \geqslant \theta(C_1) \quad \text{on } [D_3, \infty), \\ \theta(D_1) = \theta(D_2) = 0 \end{cases} \tag{3.4}$$

and

$$\begin{cases} \theta'(u) > 0 \quad \text{on } (0, C_1), \quad \theta'(u) < 0 \quad \text{on } (C_1, C_2), \quad \theta'(u) > 0 \quad \text{on } (C_2, \infty), \\ \theta'(C_1) = \theta'(C_2) = 0. \end{cases} \tag{3.5}$$

By (3.1), (3.4) and (3.5), we obtain that

$$T'(\alpha) > 0 \quad \text{for } \alpha \in (0, C_1], \qquad T'(\alpha) < 0 \quad \text{for } \alpha \in [D_1, C_2] \quad \text{and} \quad T'(\alpha) > 0 \quad \text{for } \alpha \in [D_3, \infty).$$

Hence $T(\alpha)$ has at least two critical points, a local maximum on (C_1, D_1) and a local minimum on (C_2, D_3). We then prove that $T(\alpha)$ has exactly one critical point, a local maximum on (C_1, D_1), and $T(\alpha)$ has exactly one critical point, a local minimum on (C_2, D_3), respectively.

First, we are able to prove that $T(\alpha)$ has exactly one critical point at some α^*, a local maximum, on (C_1, D_1). We show this by showing that

$$T''(\alpha) + \frac{M_1}{p\alpha}T'(\alpha) < 0 \quad \text{for } \alpha \in (C_1, D_1)$$

for some positive function

$$M_1 = \max_{0 \leqslant u \leqslant \alpha} \frac{\alpha f(\alpha) - u f(u)}{F(\alpha) - F(u)}.$$

This proof is similar to that of [14, Theorem 1.1], and consequently, we omit it.

Next, we prove that $T(\alpha)$ has exactly one critical point, a local minimum, on (C_2, D_3). If there exist numbers $0 < C_1 < \gamma < \infty$ such that (2.6) holds, since $\phi'(u) = u\theta''(u) = u[(p-2)f'(u) - uf''(u)]$, we obtain that

$$\phi'(u) < 0 \quad \text{on } (0,\gamma), \qquad \phi'(\gamma) = 0, \qquad \phi'(u) > 0 \quad \text{on } (\gamma,\infty).$$

In addition, it is easy to compute that $\phi(0) = 0$ and $\phi(C_2) = -\theta(C_2) > 0$. So we obtain that $\phi(u)$ is strictly increasing on (C_2, D_3) and $\phi(u) < \phi(C_2)$ for $u \in (0, C_2)$. The above imply that $\phi(u) < \phi(\alpha)$ for $\alpha \in (C_2, D_3)$, $u \in (0,\alpha)$. By (3.3), we obtain $T''(\alpha) + (p/\alpha)T'(\alpha) > 0$ for $\alpha \in (C_2, D_3)$. That is, if $\alpha_* \in (C_2, D_3)$ is a critical point of $T(\alpha)$, then $T(\alpha_*)$ must be a local minimum. Thus $T(\alpha)$ has exactly one critical point at α_*, a local minimum, on (C_2, D_3).

If there exist numbers $0 < \tilde{\gamma} < C_1 < \gamma < \infty$ such that (2.7) holds, since $\phi'(u) = u\theta''(u) = u[(p-2)f'(u) - uf''(u)]$, similarly, we obtain that

$$\begin{cases} \phi'(u) > 0 \quad \text{on } (0,\tilde{\gamma}), \quad \phi'(u) < 0 \quad \text{on } (\tilde{\gamma},\gamma), \quad \phi'(u) > 0 \quad \text{on } (\gamma,\infty), \\ \phi'(\tilde{\gamma}) = 0, \quad \phi'(\gamma) = 0. \end{cases}$$

In addition, it is easy to compute that $\phi(0) = 0$, $\phi(C_1) = -\theta(C_1) < 0$ and $\phi(C_2) = -\theta(C_2) > 0$. So we obtain that $\phi(u)$ is strictly increasing on (C_2, D_3) and $\phi(u) \leqslant \phi(C_2)$ for $u \in (0, C_2)$ by the assumption in (H2) that $\theta(\tilde{\gamma}) - \tilde{\gamma}\theta'(\tilde{\gamma}) - \theta(C_2) \geqslant 0$. By (3.3), we obtain $T''(\alpha) + (p/\alpha)T'(\alpha) > 0$ for $\alpha \in (C_2, D_3)$. That is, if $\alpha_* \in (C_2, D_3)$ is a critical point of $T(\alpha)$, then $T(\alpha_*)$ must be a local minimum. Thus $T(\alpha)$ has exactly one critical point at α_*, a local minimum, on (C_2, D_3).

We summarize above results and we obtain that $T(\alpha)$ has exactly two positive critical points, $\alpha^* < \alpha_*$, on $(0,\infty)$, such that $T(\alpha^*)$ is a local maximum on $(0,\infty)$ and $T(\alpha_*)$ is a local minimum on $(0,\infty)$.

So there exist

$$0 < \lambda_* \equiv \big(T(\alpha_*)\big)^p < \lambda^* \equiv \big(T\big(\alpha^*\big)\big)^p < \infty$$

such that (2.3) has exactly three positive solutions for $\lambda_* < \lambda < \lambda^*$, exactly two positive solutions for $\lambda = \lambda_*$ and $\lambda = \lambda^*$, and a unique positive solution for $0 < \lambda < \lambda_*$ and $\lambda^* < \lambda < \infty$.

The proof of Lemma 2.2 is complete. □

Proof of Lemma 2.3. First, we know that $\theta(0) = 0$, $\theta'(0) = p - 1 > 0$, and $\lim_{u\to\infty}\theta(u) = \infty$.

(i) We prove $\theta(a) < 0$. The proof is divided into two cases:

(A) $1 < p < 2$, and
(B) $p \geqslant 2$.

Proof of Case (A) $\mathbf{1 < p < 2}$. Let $a \geqslant \tilde{a} = \max\{4, (4p-2)(\ln p) + 0.7\}$.

We first consider the graph of the function

$$g(u) \equiv pf(u) = p\exp\left(\frac{au}{a+u}\right),$$

on $(0, a)$. It is easy to see that $g(u)$ satisfies

$$g(u) > 0 \quad \text{on } (0,a), \tag{3.6}$$

$$g'(u) = \frac{pa^2}{(a+u)^2}\exp\left(\frac{au}{a+u}\right) > 0 \quad \text{on } (0,a), \tag{3.7}$$

and

$$g''(u) = \frac{pa^2[a(a-2) - 2u]}{(a+u)^4}\exp\left(\frac{au}{a+u}\right) > 0 \quad \text{on } (0,a) \tag{3.8}$$

for $a \geqslant \tilde{a} \geqslant 4$.

Secondly, define $a^* \in (0, a)$ by

$$a^* = \frac{a(a - 2\ln p)}{a + 2\ln p} = a - \frac{4a\ln p}{a + 2\ln p},$$

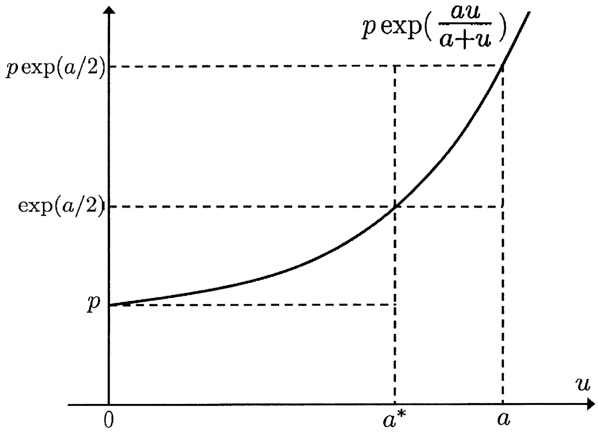

Fig. 3. Graph of function $g(u) \equiv pf(u) = p\exp(\frac{au}{a+u})$ on $(0,a)$ for $a \geqslant \tilde{a}$ and $1 < p < 2$. Note that $g''(u) > 0$ on $(0,a)$.

which satisfies

$$g(a^*) = pf(a^*) = p\exp\left(\frac{aa^*}{a+a^*}\right) = \exp(a/2).$$

We then compute that

$$\begin{aligned}
\theta(a) &= pF(a) - af(a) \\
&= \int_0^a p\exp\left(\frac{at}{a+t}\right)dt - a\exp(a/2) \\
&= \int_{a^*}^a \left[p\exp\left(\frac{at}{a+t}\right) - \exp(a/2)\right]dt + \int_0^{a^*}\left[p\exp\left(\frac{at}{a+t}\right) - \exp(a/2)\right]dt \\
&= \int_{a^*}^a \left[p\exp\left(\frac{at}{a+t}\right) - \exp(a/2)\right]dt - \int_0^{a^*}\left[\exp(a/2) - p\exp\left(\frac{at}{a+t}\right)\right]dt \\
&< \frac{1}{2}\left\{\frac{4a\ln p}{a+2\ln p}(p-1)\exp(a/2) - \left[\exp(a/2) - p\right]\frac{a(a-2\ln p)}{a+2\ln p}\right\}
\end{aligned}$$

since

$$\int_{a^*}^a \left[p\exp\left(\frac{at}{a+t}\right) - \exp(a/2)\right]dt < \frac{1}{2}(p-1)\exp(a/2)\left(a-a^*\right) = \frac{1}{2}\left\{\frac{4a\ln p}{a+2\ln p}(p-1)\exp(a/2)\right\}$$

and

$$-\int_0^{a^*}\left[\exp(a/2) - p\exp\left(\frac{at}{a+t}\right)\right]dt < -\frac{1}{2}\left[\exp(a/2) - p\right]a^* = -\frac{1}{2}\left[\exp(a/2) - p\right]\frac{a(a-2\ln p)}{a+2\ln p}$$

by (3.6)–(3.8), see Fig. 3. So we have

$$\begin{aligned}
\theta(a) &< \frac{1}{2}\left\{\frac{4a\ln p}{a+2\ln p}(p-1)\exp(a/2) - \left[\exp(a/2) - p\right]\frac{a(a-2\ln p)}{a+2\ln p}\right\} \\
&= -\frac{a}{2(a+2\ln p)}\left\{(a-2\ln p)\left[\exp(a/2) - p\right] - (4p-4)(\ln p)\exp(a/2)\right\}
\end{aligned}$$

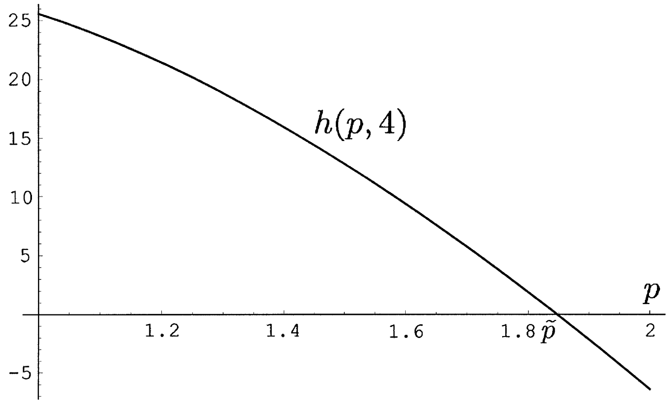

Fig. 4. Graph of function $h(p,4)$ for $1<p<2$. $h(\tilde{p},4)=0$ where $\tilde{p}\approx 1.847$.

$$\begin{aligned}
&= -\frac{a}{2(a+2\ln p)}\bigl\{\bigl[a-(4p-2)\ln p\bigr]\exp(a/2)-p(a-2\ln p)\bigr\} \\
&\equiv -\frac{a}{2(a+2\ln p)}h(p,a)
\end{aligned} \tag{3.9}$$

where

$$h(p,a)=\bigl[a-(4p-2)\ln p\bigr]\exp(a/2)-p(a-2\ln p).$$

So

$$\theta(a)<0 \quad \text{if } h(p,a)>0.$$

For fixed $1<p<2$ and for $a\geqslant \tilde{a}\equiv \max\{4,(4p-2)(\ln p)+0.7\}$, we compute that

$$\begin{aligned}
\frac{\partial}{\partial a}h(p,a) &= \left[\frac{a}{2}-(2p-1)(\ln p)+1\right]\exp(a/2)-p \\
&\geqslant \bigl[(2p-1)(\ln p)+0.35-(2p-1)(\ln p)+1\bigr]p-p
\end{aligned}$$

since $a\geqslant\tilde{a}\geqslant(4p-2)(\ln p)+0.7$ and $\exp(a/2)\geqslant\exp((2p-1)\ln p)>\exp(\ln p)=p$. So

$$\frac{\partial}{\partial a}h(p,a)\geqslant\bigl[(2p-1)(\ln p)+0.35-(2p-1)(\ln p)+1\bigr]p-p=0.35p>0.$$

In addition,

(I) For $1<p\leqslant p^*$ (≈ 1.846), $\tilde{a}(p)=4$. Letting $a=\tilde{a}=4$, we obtain

$$h(p,4)=\bigl[4-(4p-2)\ln p\bigr]\exp(2)-p(4-2\ln p)>0,$$

see Fig. 4. (Note that the first positive zero for $h(p,4)$ is $\tilde{p}\approx 1.847$.)

(II) For $(1.846\approx)$ $p^*<p<2$, $\tilde{a}(p)=(4p-2)(\ln p)+0.7$. Letting $a=\tilde{a}(p)=(4p-2)(\ln p)+0.7$, we compute that

$$\begin{aligned}
h(p,\tilde{a}(p)) &= \bigl[a-(4p-2)\ln p\bigr]\exp(a/2)-p(a-2\ln p) \\
&= \bigl[(4p-2)(\ln p)+0.7-(4p-2)(\ln p)\bigr]\exp\bigl[(2p-1)(\ln p)+0.35\bigr] \\
&\quad -p\bigl[(4p-2)(\ln p)+0.7-2\ln p\bigr] \\
&= 0.7p^{2p-1}\exp(0.35)-4(p-1)p(\ln p)-0.7p \\
&> 0,
\end{aligned}$$

see Fig. 5. (Note that $h(\bar{p},\tilde{a}(\bar{p}))=0$ for $\bar{p}\approx 1.828$ and $h(p,\tilde{a}(p))>0$ for $p>p^*\approx 1.846$.)

So by above, we obtain that, if $1<p<2$ and $a\geqslant\tilde{a}$, then $h(p,a)>0$, and hence $\theta(a)<0$. So Lemma 2.3(i) holds for Case (A) $1<p<2$. $\square$

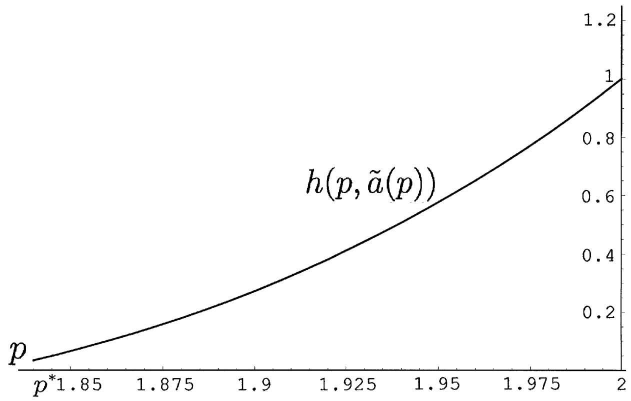

Fig. 5. Graph of function $h(p, \tilde{a}(p))$ for $(1.846 \approx)\ p^* < p < 2$.

Proof of Case (B) $p \geqslant 2$. Let $a \geqslant \tilde{a} = (4p-2)(\ln p) + \eta$.

Similarly as before, we first consider the graph of the function

$$g(u) \equiv pf(u) = p \exp\left(\frac{au}{a+u}\right)$$

on $(0, a)$. It is easy to see that $g(u)$ satisfies (3.6)–(3.8) for $a \geqslant \tilde{a} > 4$ by (2.2).

Secondly, define $a^* \in (0, a)$ by

$$a^* = \frac{a(a - 2\ln p)}{a + 2\ln p} = a - \frac{4a \ln p}{a + 2\ln p},$$

which satisfies

$$g(a^*) = pf(a^*) = p \exp\left(\frac{aa^*}{a+a^*}\right) = \exp(a/2).$$

Then, by the same arguments as we did in the proof of Lemma 2.3(i), we obtain (3.9)

$$\theta(a) < -\frac{a}{2(a+2\ln p)} h(p, a)$$

where

$$h(p, a) = \left[a - (4p-2)\ln p\right] \exp(a/2) - p(a - 2\ln p).$$

So

$$\theta(a) < 0 \quad \text{if } h(p, a) > 0.$$

For fixed $p \geqslant 2$ and for $a \geqslant \tilde{a} = (4p-2)(\ln p) + \eta$, we compute that

$$\begin{aligned}
\frac{\partial}{\partial a} h(p, a) &= \left[\frac{a}{2} - (2p-1)(\ln p) + 1\right] \exp(a/2) - p \\
&\geqslant \left[(2p-1)(\ln p) + \frac{\eta}{2} - (2p-1)(\ln p) + 1\right] p - p
\end{aligned}$$

since $a \geqslant \tilde{a} = (4p-2)(\ln p) + \eta$ and $\exp(a/2) \geqslant \exp((2p-1)\ln p) > \exp(\ln p) = p$. So

$$\frac{\partial}{\partial a} h(p, a) \geqslant \left[(2p-1)(\ln p) + \frac{\eta}{2} - (2p-1)(\ln p) + 1\right] p - p = \frac{\eta p}{2} > 0. \tag{3.10}$$

In addition, for $p \geqslant 2$, letting $a = \tilde{a} = (4p-2)(\ln p) + \eta$, we obtain

$$\begin{aligned} h\big(p, (4p-2)(\ln p) + \eta\big) &= \eta \exp\big((2p-1)(\ln p) + \eta/2\big) - p\big[(4p-2)(\ln p) + \eta - 2\ln p\big] \\ &= \eta \exp\big((2p-1)(\ln p) + \eta/2\big) - p\big[(4p-4)(\ln p) + \eta\big] \\ &= p\big[\eta \exp(\eta/2) p^{2p-2} - (4p-4)(\ln p) + \eta\big] \\ &\equiv pR(p), \end{aligned}$$

where

$$R(p) = \eta \exp(\eta/2) p^{2p-2} - (4p-4)(\ln p) + \eta.$$

We compute that

$$\begin{aligned} R'(p) &= 2\eta \exp(\eta/2) p^{2p-2}\left[\frac{p-1}{p} + \ln p\right] - \frac{4p-4}{p} - 4(\ln p) \\ &= 2\big[\eta \exp(\eta/2) p^{2p-2} - 2\big]\left[\frac{p-1}{p} + \ln p\right] \\ &> 0 \quad \text{on } [2, \infty) \end{aligned}$$

since for $p \geqslant 2$,

(I) $\eta \exp(\eta/2) p^{2p-2} - 2 > 0$ since $\eta \exp(\eta/2) p^{2p-2} - 2$ is strictly increasing in $p > 2$ and $\eta \exp(\eta/2) 2^2 - 2 \approx 1.394 > 0$, and
(II) $\frac{p-1}{p} + \ln p > 0$.

In addition,

$$R(2) = 4\eta \exp(\eta/2) - 4(\ln 2) + \eta \approx 1.244 > 0.$$

So by above, we conclude that, for $p \geqslant 2$, $R(p) > 0$ and hence $h(p, (4p-2)(\ln p) + \eta) > 0$. Thus, by (3.10), for $a \geqslant \tilde{a} = (4p-2)(\ln p) + \eta$, we have $h(p, a) > 0$ and hence $\theta(a) < 0$. So Lemma 2.3(i) holds for Case (B) $p \geqslant 2$. □

By above, Lemma 2.3(i) holds for $p > 1$.

(ii) For $1 < p < 2$, by (2.15), $\theta(u)$ has exactly two positive critical points at $C_1 < C_2$, and we compute that

$$C_1 = \frac{a}{2(p-1)}\big[a - 2p + 2 - \sqrt{a^2 + 4a - 4pa}\big],$$
$$C_2 = \frac{a}{2(p-1)}\big[a - 2p + 2 + \sqrt{a^2 + 4a - 4pa}\big].$$

Also, by (2.16), it is easy to see that $\theta(u)$ has exactly one positive inflection point at γ, and we compute that

$$\gamma = \frac{a}{2p}\big[a - 2p + 2 + \sqrt{a^2 + 4a - 4pa + 4}\big] > \frac{a}{2p}\big[\big(4(p-1) - 2p\big) + 2 + 2\big]$$

since $a^2 + 4a - 4pa + 4 = a[a - 4(p-1)] + 4 > 4$ and $a > 4(p-1)$ by (2.2). So

$$\gamma > \frac{a}{2p}\big[\big(4(p-1) - 2p\big) + 2 + 2\big] = a.$$

Since $\theta(a) < 0$ and by above analysis for $\theta(u)$, we know that $\theta(u)$ has exactly two positive zeros at $D_1 < D_2$ such that $0 < C_1 < D_1 < \gamma < C_2 < D_2 < \infty$. So Lemma 2.3(ii) holds.

(iii) For $p \geqslant 2$, similarly, by (2.15), $\theta(u)$ has exactly two positive critical points at $C_1 < C_2$, and we compute that

$$C_1 = \frac{a}{2(p-1)}\big[a - 2p + 2 - \sqrt{a^2 + 4a - 4pa}\big],$$
$$C_2 = \frac{a}{2(p-1)}\big[a - 2p + 2 + \sqrt{a^2 + 4a - 4pa}\big].$$

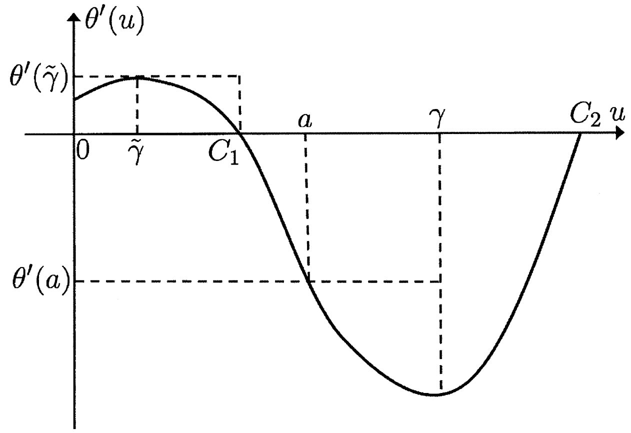

Fig. 6. Graph of function $\theta'(u)$ on $(0, C_2)$. Note that $-C_1\theta'(\tilde\gamma) - (\gamma - a)\theta'(a) > 0$ for $a \geqslant \tilde a$ and $p \geqslant 2$.

Also, by (2.16), it is easy to see that $\theta(u)$ has exactly two positive inflection points at $\tilde\gamma < \gamma$, and we compute that

$$\tilde\gamma = \frac{a}{2p}\left[a - 2p + 2 - \sqrt{a^2 + 4a - 4pa + 4}\right],$$
$$\gamma = \frac{a}{2p}\left[a - 2p + 2 + \sqrt{a^2 + 4a - 4pa + 4}\right] > \frac{a}{2p}\left[\left(4(p-1) - 2p\right) + 2 + 2\right]$$

since $a^2 + 4a - 4pa + 4 = a[a - 4(p-1)] + 4 > 4$ and $a > 4(p-1)$ by (2.2). So

$$\gamma > \frac{a}{2p}\left[\left(4(p-1) - 2p\right) + 2 + 2\right] = a.$$

Since $\theta(a) < 0$ and by above analysis for $\theta(u)$, we know that $\theta(u)$ has exactly two positive zeros at $D_1 < D_2$ such that $0 < \tilde\gamma < C_1 < D_1 < \gamma < C_2 < D_2 < \infty$.

Finally, we prove (2.13) $\theta(\tilde\gamma) - \tilde\gamma\theta'(\tilde\gamma) - \theta(C_2) > 0$ by applying Lemma 2.4, see Fig. 6.

Let $p \geqslant 2$ and $a \geqslant \tilde a = (4p - 2)(\ln p) + \eta$. Suppose Lemma 2.4 holds. We then observe the graph of $\theta'(u)$ on $(0, C_2)$ as in Fig. 6 and we compute that

$$\begin{aligned}
\theta(\tilde\gamma) - \tilde\gamma\theta'(\tilde\gamma) - \theta(C_2) &= \int_0^{\tilde\gamma} \theta'(u)\,du - \tilde\gamma\theta'(\tilde\gamma) - \int_0^{C_2} \theta'(u)\,du \\
&= \int_0^{\tilde\gamma} \theta'(u)\,du - \tilde\gamma\theta'(\tilde\gamma) - \int_0^{C_1} \theta'(u)\,du - \int_{C_1}^{C_2} \theta'(u)\,du \\
&= -\tilde\gamma\theta'(\tilde\gamma) - \int_{\tilde\gamma}^{C_1} \theta'(u)\,du - \int_{C_1}^{C_2} \theta'(u)\,du \\
&> -C_1\theta'(\tilde\gamma) - (\gamma - a)\theta'(a)
\end{aligned}$$

since $-\tilde\gamma\theta'(\tilde\gamma) - \int_{\tilde\gamma}^{C_1} \theta'(u)\,du > -C_1\theta'(\tilde\gamma)$ and $-\int_{C_1}^{C_2} \theta'(u)\,du > -(\gamma - a)\theta'(a)$. So we have

$$\begin{aligned}
\theta(\tilde\gamma) - \tilde\gamma\theta'(\tilde\gamma) - \theta(C_2) &> -C_1\theta'(\tilde\gamma) - (\gamma - a)\theta'(a) \\
&= -C_1\left[(p-1)f(\tilde\gamma) - \tilde\gamma f'(\tilde\gamma)\right] - (\gamma - a)\theta'(a) \\
&> -C_1(p-1)f(\tilde\gamma) - (\gamma - a)\theta'(a) \quad \left(\text{since } \tilde\gamma f'(\tilde\gamma) > 0\right) \\
&> -\frac{43}{100}a(p-1)f\left(\frac{3}{20}a\right) - \frac{39}{100}a\theta'(a)
\end{aligned}$$

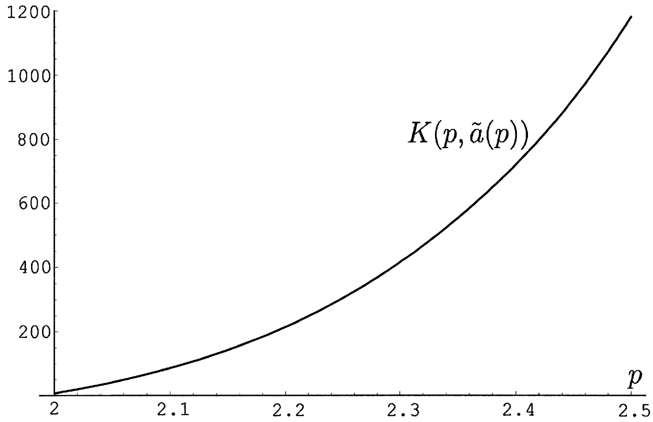

Fig. 7. Graph of function $K(p, \tilde{a}(p))$ for $2 \leqslant p \leqslant 2.5$. $K(2, \tilde{a}(2)) \approx 6.183 > 0$.

by Lemma 2.4(i)–(iii) and since f is increasing on $(0, a)$. So

$$\begin{aligned}\theta(\tilde{\gamma}) - \tilde{\gamma}\theta'(\tilde{\gamma}) - \theta(C_2) &> -\frac{43}{100}a(p-1)f\left(\frac{3}{20}a\right) - \frac{39}{100}a\theta'(a)\\ &= -\frac{43}{100}a(p-1)\exp(3a/23) + \frac{39}{100}a\left(\frac{a}{4} - p + 1\right)\exp(a/2)\\ &\equiv \frac{1}{400}a\exp(3a/23)K(p,a), \end{aligned} \tag{3.11}$$

where

$$K(p,a) = 39(a - 4p + 4)\big[\exp(17a/46)\big] - 172(p-1).$$

We easily compute that, for $a \geqslant \tilde{a} = (4p-2)(\ln p) + \eta$,

$$\frac{\partial}{\partial a}K(p,a) = \frac{39}{46}(17a - 68p + 114)\big[\exp(17a/46)\big] > 0$$

since $17a - 68p + 114 \geqslant 17[(4p-2)(\ln p) + \eta] - 68p + 114 > 0$ for $p \geqslant 2$, we omit the proof. In addition,

$$K\big(p, \tilde{a}(p)\big) = 39\big[(4p-2)(\ln p) + \eta - 4p + 4\big]\big[\exp\big(17\big((4p-2)(\ln p) + \eta\big)/46\big)\big] - 172(p-1) > 0$$

by some numerical simulation given in Fig. 7, which shows that $K(p, \tilde{a}(p))$ is a strictly increasing function of $p \geqslant 2$ and $K(2, \tilde{a}(2)) \approx 6.183 > 0$.

We summarize above results and we conclude that, for $p \geqslant 2$ and $a \geqslant \tilde{a} = (4p-2)(\ln p) + \eta$, $K(p,a) > 0$, and hence $\theta(\tilde{\gamma}) - \tilde{\gamma}\theta'(\tilde{\gamma}) - \theta(C_2) > 0$ by (3.11). This completes the proof of Lemma 2.3(iii).

The proof of Lemma 2.3 is complete. □

Proof of Lemma 2.4. Let $p \geqslant 2$.

(i) We take

$$R_1(p,a) \equiv \frac{\tilde{\gamma}}{a} = \frac{1}{2p}\big[a - 2p + 2 - \sqrt{a^2 + 4a - 4pa + 4}\big].$$

If $a \geqslant \tilde{a} = (4p-2)(\ln p) + \eta$, then it is easy to compute that

$$\frac{\partial}{\partial a}R_1(p,a) = \frac{-a + 2p - 2 + \sqrt{a^2 + 4a - 4pa + 4}}{2p\sqrt{a^2 + 4a - 4pa + 4}} \begin{cases} < 0 & \text{if } p > 2,\\ = 0 & \text{if } p = 2\end{cases}$$

since $a > 2p - 2$ by (2.2), and

$$(-a + 2p - 2)^2 - \big(a^2 + 4a - 4pa + 4\big) = 4p(p-2)\begin{cases} > 0 & \text{if } p > 2,\\ = 0 & \text{if } p = 2.\end{cases}$$

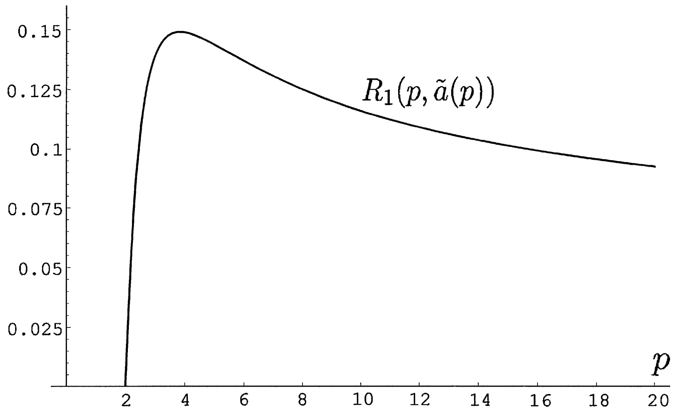

Fig. 8. Graph of function $R_1(p, \tilde{a}(p))$ for $2 \leqslant p \leqslant 20$. $\max_{p \geqslant 2} R_1(p, \tilde{a}(p)) = R_1(p_1, \tilde{a}(p_1)) \approx 0.149$ for $p = p_1 \approx 3.838$.

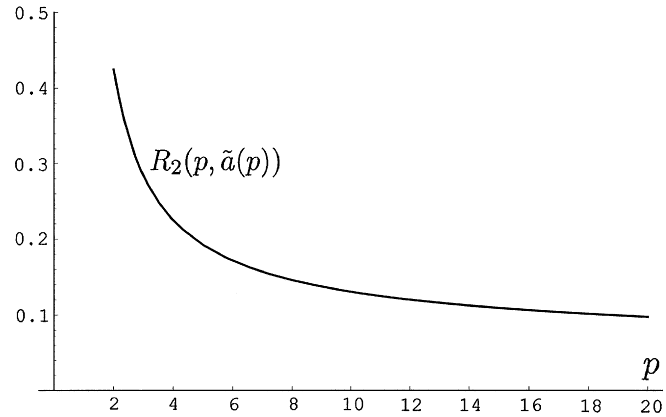

Fig. 9. Graph of function $R_2(p, \tilde{a}(p))$ for $2 \leqslant p \leqslant 20$. $\max_{p \geqslant 2} R_2(p, \tilde{a}(p)) = R_2(2, \tilde{a}(2)) \approx 0.4243$.

Then, for all $p \geqslant 2$ and $a \geqslant \tilde{a}(p)$, we obtain that

$$R_1(p, a) \leqslant R_1\big(p, \tilde{a}(p)\big) \leqslant \max_{p \geqslant 2} R_1\big(p, \tilde{a}(p)\big) = R_1\big(p_1, \tilde{a}(p_1)\big) \approx 0.149 < 0.150 = \frac{3}{20}$$

for $p_1 \approx 3.838$, by some numerical simulation given in Fig. 8. So part (i) follows.

(ii) We take

$$R_2(p, a) \equiv \frac{C_1}{a} = \frac{1}{2(p-1)}\left[a - 2p + 2 - \sqrt{a^2 + 4a - 4pa}\right].$$

If $a \geqslant \tilde{a} = (4p-2)(\ln p) + \eta$, then it is easy to compute that

$$\frac{\partial}{\partial a} R_2(p, a) = \frac{-a + 2p - 2 + \sqrt{a^2 + 4a - 4pa}}{2(p-1)\sqrt{a^2 + 4a - 4pa}} < 0$$

since $a > 2p - 2$ by (2.2), and

$$(-a + 2p - 2)^2 - \big(a^2 + 4a - 4pa\big) = 4(p-1)^2 > 0.$$

Then, for all $p \geqslant 2$ and $a \geqslant \tilde{a}(p)$, we obtain that

$$R_2(p, a) \leqslant R_2\big(p, \tilde{a}(p)\big) \leqslant \max_{p \geqslant 2} R_2\big(p, \tilde{a}(p)\big) = R_2\big(2, \tilde{a}(2)\big) \approx 0.424 < 0.430 = \frac{43}{100}$$

by some numerical simulation given in Fig. 9. (Numerical simulation shows that $R_2(p, \tilde{a}(p))$ is a strictly decreasing function of $p \geqslant 2$.) So part (ii) follows.

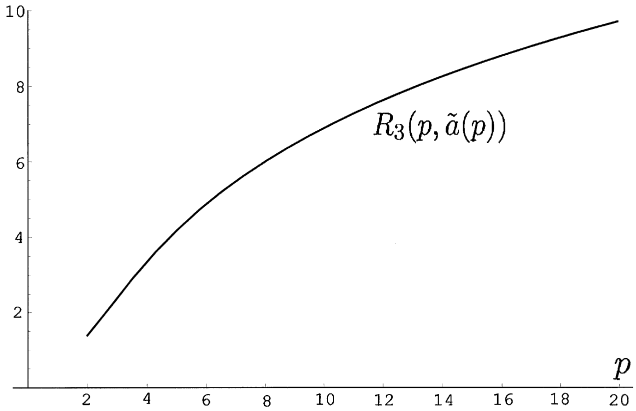

Fig. 10. Graph of function $R_3(p, \tilde{a}(p))$ for $2 \leqslant p \leqslant 20$. $\min_{p \geqslant 2} R_3(p, \tilde{a}(p)) = R_3(2, \tilde{a}(2)) \approx 1.390$.

(iii) We take

$$R_3(p, a) \equiv \frac{\gamma}{a} = \frac{1}{2p}\left[a - 2p + 2 + \sqrt{a^2 + 4a - 4pa + 4}\right].$$

If $a \geqslant \tilde{a} = (4p - 2)(\ln p) + \eta$, then it is easy to compute that

$$\frac{\partial}{\partial a} R_3(p, a) = \frac{a - 2p + 2 + \sqrt{a^2 + 4a - 4pa + 4}}{2p\sqrt{a^2 + 4a - 4pa + 4}} > 0 \quad \text{if } p \geqslant 2$$

since $a > 2p - 2$ by (2.2). Then, for all $p \geqslant 2$ and $a \geqslant \tilde{a}(p)$, we obtain that

$$R_3(p, a) \geqslant R_3\bigl(p, \tilde{a}(p)\bigr) \geqslant \min_{p \geqslant 2} R_3\bigl(p, \tilde{a}(p)\bigr) = R_3\bigl(2, \tilde{a}(2)\bigr) \approx 1.3904 > 1.390 = \frac{139}{100}$$

by some numerical simulation given in Fig. 10. (Numerical simulation shows that $R_3(p, \tilde{a}(p))$ is a strictly increasing function of $p \geqslant 2$.) So part (iii) follows.

The proof of Lemma 2.4 is complete. □

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